

Optimal control of nonlinear systems governed by Dirichlet fractional Laplacian in the minimax framework

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Abstract

We consider an optimal control problem governed by a class of boundary value problem with the Dirichlet fractional Laplacian. Some sufficient condition for the existence of optimal processes is stated. The proof of the main result relies on variational structure of the problem. To show that boundary value problem with the Dirichlet fractional Laplacian has a weak solution we employ the renowned Ky Fan Theorem.

Keywords: optimal control, fractional Laplacian, variational methods, saddle points, stability, Kuratowski-Painlevé limit

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ for $n \geq 3$ be a bounded domain with a Lipschitz boundary. In this paper we consider a boundary value problem for nonlinear nonlocal vector equation of the form

$$(1) \quad (-\Delta)^{\alpha/2} \psi(x) + f(x, \psi(x), w(x)) = 0 \text{ in } \Omega,$$

$$(2) \quad \psi(x) = 0 \text{ on } \partial\Omega$$

where a vector function ψ belongs to some fractional Sobolev space $H_0^{\alpha/2}$ and a control w belongs to L^p for $\alpha \in (1, 2)$. The problems involving different notions of the fractional Laplacian attracted in the recent years a lot of attention motivated by the problems in finances [1], mechanics [7, 9, 16], hydrodynamics [10, 17, 19, 41, 42], elastostatics [7, 16] or probability [1, 9, 20]. It should be moreover noted that at least two notions of fractional Laplace operator coexist: the first the Dirichlet fractional Laplacian defined by the spectral properties of the Dirichlet Laplace operator, see [5, 15] and the second one defined via the singular integral or the infinitesimal generator of the Lévy semigroup, for a list of relevant references, see [1, 4, 8, 9, 20, 43]. In this paper we use the Dirichlet fractional Laplacian set in the spectral framework. The problems governed by the Dirichlet fractional Laplacian can be seen as a natural extensions of the problems discussed in [11, 12, 44, 45] involving the standard Laplace operator. Specifically, we focus our attention on the continuous dependence of the solutions on the functional parameters and then on the existence of the optimal solutions minimizing some cost functional. For related results concerning optimal solution we refer the interested readers, for example, to papers [7, 45]. The framework requires the minimax geometry (cf. [32, 36, 40, 46]) for concave-convex functionals of action allowing by Ky Fan Theorem the existence of saddle point solutions. For related results involving some notions of the fractional Laplacian, see, among others, papers [14, 24, 25, 38] with the minimax geometry setting.

To be more specific we shall consider a control problem governed by a system of nonlinear fractional differential equations in Ω

$$(3) \quad \begin{cases} -(-\Delta)^{\alpha/2}u(x) + G_u(x, u(x), v(x), w(x)) = 0 \\ (-\Delta)^{\alpha/2}v(x) + G_v(x, u(x), v(x), w(x)) = 0 \end{cases}$$

with the boundary data

$$(4) \quad u(x) = 0, v(x) = 0 \text{ on } \partial\Omega.$$

Clearly, problem (3) – (4) is a particular case of (1) – (2) with $\psi = (-u, v)$ and $f = (G_u, G_v)$. We prove in Section 3 that control problem (3) – (4) possesses at least one weak solution for any control w . The results concerning the continuous dependence of weak solution on controls are discussed in Section 4. Without going into details, for a given control w_k , denote by (u_k, v_k) a weak solution of problem (3) – (4), then if the sequence $\{w_k\}_{k \in \mathbb{N}}$ tends to w_0 in appropriate topology of L^p , then the sequence $\{(u_k, v_k)\}_{k \in \mathbb{N}}$ tends to (u_0, v_0) in the strong topology of $H_0^{\alpha/2} \times H_0^{\alpha/2}$. In other words, we have proved that boundary value problem (3) – (4) is well-posed, i.e. the solution exists and it continuously depends on controls. Section 5 is devoted to the investigation of optimal control problem. The proof of the existence of the optimal solution, which is the main result of the paper, relies on the continuous dependence results from Section 4. Finally, some examples are presented.

2 Statement of the problem

Throughout the paper, we shall assume that $\Omega \subset \mathbb{R}^n$ with $n > 2$ is a bounded domain with a Lipschitz boundary i.e. $\Omega \in C^{0,1}$, see [30]. Moreover, we shall use spectral properties of the fractional Laplacian in the case of bounded domain Ω with smooth boundary. The powers $(-\Delta)^{\alpha/2}$ of the positive Laplace operator $(-\Delta)$, in a bounded domain with zero Dirichlet boundary data are defined through the spectral decomposition using the powers of the eigenvalues of the original operator. Let (z_k, ρ_k) for $k \in \mathbb{N}$ be the system of the eigenfunctions and eigenvalues of the Laplace operator $(-\Delta)$ on Ω with the homogeneous Dirichlet condition on $\partial\Omega$. Then $(z_k, \rho_k^{\alpha/2})$ for $k \in \mathbb{N}$ is the system of the eigenfunctions and eigenvalues of the fractional Laplacian $(-\Delta)^{\alpha/2}$ on Ω , also with the homogeneous boundary Dirichlet condition. By $H_0^{\alpha/2}(\Omega)$, we can denote the space of functions $z = z(x)$ defined on a bounded, smooth domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, such that $z = \sum_{k=1}^{\infty} a_k z_k$ and $\sum_{k=1}^{\infty} a_k^2 \rho_k^{\alpha/2} < \infty$, with the norm defined by the formula

$$\|z\|_{H_0^{\alpha/2}(\Omega)}^2 = \sum_{k=1}^{\infty} a_k^2 \rho_k^{\alpha/2} = \left\| (-\Delta)^{\alpha/4} z \right\|_{L^2(\Omega)}^2,$$

see [22, Proposition 4.4]. Moreover, the fractional Laplacian acts on $z = \sum_{k=1}^{\infty} a_k z_k \in H_0^{\alpha/2}(\Omega)$ as

$$(-\Delta)^{\alpha/2} z = \sum_{k=1}^{\infty} a_k \rho_k^{\alpha/2} z_k.$$

There exists also a different notion of the fractional Laplacian, defined via singular integral on the whole of \mathbb{R}^n as

$$(-\Delta)^{\alpha/2} z(x) = -\frac{1}{2} \int_{\mathbb{R}^n} \frac{z(x+y) + z(x-y) - 2z(x)}{|x-y|^{n+\alpha}} dy$$

for all $x \in \mathbb{R}^n$ which can be restricted to the functions with some values on Ω and zero value outside the set Ω . It should be underlined, however, that it leads to nonequivalent definition and therefore is often referred to as the restricted fractional Laplacian as in [10, 23, 37] not to be confused with the spectral Dirichlet fractional

Laplacian used in this paper. For the differences between two notions of the fractional Laplacian one can see, for example, [4, 9, 20, 39], where the spectral analysis of both the operators were carried over.

It is worth reminding the reader that for a bounded domain with a Lipschitz boundary, the fractional Sobolev space $H^{\alpha/2}(\Omega)$ is compactly embedded into $L^s(\Omega)$ for $s \in [1, 2_\alpha^*)$ where $2_\alpha^* = 2n/(n - \alpha)$ for $n > 2$ and the inequality

$$\|z\|_{L^s(\Omega)} \leq C \|z\|_{H^{\alpha/2}(\Omega)}$$

holds, cf. [22, Corollary 7.2].

In what follows we shall also use the following result.

Remark 2.1 *The principal eigenvalue ρ_1 of Laplacian appears in the inequality*

$$\rho_1^{\alpha/2} \leq \inf \left\{ \frac{\int_\Omega |(-\Delta)^{\alpha/4} u(x)|^2 dx}{\int_\Omega |u(x)|^2 dx}; u \in H_0^{\alpha/2}(\Omega), u \neq 0 \right\}.$$

Indeed, $(-\Delta)^{\alpha/4} u_1 = \rho_1^{\alpha/4} u_1$, so infimum on the right hand side of the above inequality is greater or equal to $\rho_1^{\alpha/2}$. Moreover, the infimum is attained, since $\|u\|_{H_0^{\alpha/2}(\Omega)}^2 = \int_\Omega |(-\Delta)^{\alpha/4} u(x)|^2 dx$ is weakly lower semicontinuous, convex and coercive as the norm in the reflexive space, for details see [3, 26, 27].

To obtain the fractional Poincaré inequality of the form

$$(5) \quad \rho_1^{\alpha/2} \int_\Omega |u(x)|^2 dx \leq \int_\Omega \left| (-\Delta)^{\alpha/4} u(x) \right|^2 dx,$$

we could apply either straightforward spectral analysis as in Remark 2.1 or the following theorem with $F(t) = t^{\alpha/2}$.

Theorem 2.1 ([31, Theorem 2.8]) *Let F be a continuous, increasing and polynomially bounded real-valued functional on $[0, \infty)$, in particular, $F(t) > 0$ for $t > 0$. Then we have the following fractional order Poincaré inequality*

$$F(\sqrt{\rho_1}) \|u\|_{L^2} \leq \left\| F\left(\sqrt{-\Delta}\right) u \right\|_{L^2}.$$

For the fractional Poincaré inequality with general measures involving nonlocal quantities on unbounded domain see also paper [33] by Mouhot et al..

Remark 2.2 *The fractional Sobolev inequality extending Poincaré inequality to $L^s(\Omega)$ with, in general, not optimal constant $C > 0$, has the form*

$$\int_\Omega \left| (-\Delta)^{\alpha/4} u(x) \right|^2 dx \geq C \left(\int_\Omega |u(x)|^s dx \right)^{\frac{2}{s}}$$

for any $s \in [1, 2_\alpha^*)$, $n > \alpha$, and every $u \in H_0^{\alpha/2}(\Omega)$. When $s = 2_\alpha^*$ the best constant in the fractional Sobolev inequality will be denoted by $S(\alpha, n)$. This constant is explicit and independent of the domain, its exact value is

$$S(\alpha, n) = \frac{2\pi^{\frac{n}{2}} \Gamma(\frac{n+\alpha}{2}) \Gamma(\frac{2-\alpha}{2}) (\Gamma(\frac{n}{2}))^{\frac{\alpha}{n}}}{\Gamma(\frac{n}{2}) \Gamma(\frac{n-\alpha}{2}) (\Gamma(n))^{\frac{\alpha}{2}}}$$

where Γ is the standard Euler Gamma function defined by $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt$, compare [5, pg. 6138]. When $s = 2$ we recover the fractional Poincaré inequality without optimal constant in general.

In this paper we consider systems of nonlinear fractional differential equations of the form

$$(6) \quad \begin{cases} -(-\Delta)^{\alpha/2}u(x) + G_u(x, u(x), v(x), w(x)) = 0 & \text{in } \Omega \\ -(-\Delta)^{\alpha/2}v(x) + G_v(x, u(x), v(x), w(x)) = 0 & \text{in } \Omega \\ u(x) = 0, v(x) = 0 & \text{on } \partial\Omega \end{cases}$$

where $u \in H_0^{\alpha/2}(\Omega)$, $v \in H_0^{\alpha/2}(\Omega)$, G is a scalar function defined on the set $\Omega \times \mathbb{R}^{2+m}$ and $w \in \mathcal{W}$ with

$$\mathcal{W} = \{w \in L^p(\Omega, \mathbb{R}^m) : w(x) \in M \text{ for a.e. } x \in \Omega\}$$

where $M \subset \mathbb{R}^m$ is convex and bounded. The set \mathcal{W} will be referred to as a set of distributed parameters or distributed controls.

We shall investigate the question of the continuous dependence on control $w \in \mathcal{W}$ of weak solutions of problem (6) in the space $\mathbb{H}_0^{\alpha/2} = H_0^{\alpha/2}(\Omega) \times H_0^{\alpha/2}(\Omega)$. We replace this question, under some assumption about the function $G = G(x, u, v, w)$, with the question of the continuous dependence on controls of saddle points of the functional of action for problem (6) of the form

$$(7) \quad F_w(u, v) = \int_{\Omega} \left(\frac{1}{2} \left| (-\Delta)^{\alpha/4}v(x) \right|^2 - \frac{1}{2} \left| (-\Delta)^{\alpha/4}u(x) \right|^2 + G(x, u(x), v(x), w(x)) \right) dx,$$

defined on the space $\mathbb{H}_0^{\alpha/2}$. The space $\mathbb{H}_0^{\alpha/2}$ will be considered with the norm

$$\|(u, v)\|_{\mathbb{H}_0^{\alpha/2}}^2 = \|u\|_{H_0^{\alpha/2}(\Omega)}^2 + \|v\|_{H_0^{\alpha/2}(\Omega)}^2.$$

Let us recall that a pair $(u_0, v_0) \in \mathbb{H}_0^{\alpha/2}$ is a saddle point of a functional $F_w : \mathbb{H}_0^{\alpha/2} \rightarrow \mathbb{R}$ if

$$F_w(u, v_0) \leq F_w(u_0, v_0) \leq F_w(u_0, v)$$

for any $u \in H_0^{\alpha/2}(\Omega)$ and $v \in H_0^{\alpha/2}(\Omega)$ which is equivalent to

$$\sup_u \inf_v F_w(u, v) = \inf_v \sup_u F_w(u, v) = F_w(u_0, v_0)$$

provided that $\sup_u \inf_v F_w(u, v)$ and $\inf_v \sup_u F_w(u, v)$ are finite and attainable. Moreover a pair $(u, v) \in \mathbb{H}_0^{\alpha/2}$ is the the weak solution of problem (6) if, for any $(g, h) \in \mathbb{H}_0^{\alpha/2}$, the following equalities hold

$$\begin{cases} -\int_{\Omega} (-\Delta)^{\alpha/4}u(x) (-\Delta)^{\alpha/4}g(x) dx + \int_{\Omega} G_u(x, u(x), v(x), w(x)) g(x) dx = 0 & \text{in } \Omega, \\ \int_{\Omega} (-\Delta)^{\alpha/4}v(x) (-\Delta)^{\alpha/4}h(x) dx + \int_{\Omega} G_v(x, u(x), v(x), w(x)) h(x) dx = 0 & \text{in } \Omega, \end{cases}$$

compare with [5, Definition 2.1].

Let us make the following assumptions:

(A1) G, G_u, G_v are Carathéodory functions, i.e. they are measurable with respect to x for any $(u, v, w) \in \mathbb{R}^{2+m}$ and continuous with respect to (u, v, w) for a.e. $x \in \Omega$;

(A2) for $p = \infty$, there exists $c > 0$ such that

$$\begin{aligned} |G(x, u, v, w)| &\leq c(1 + |u|^s + |v|^s), \\ |G_u(x, u, v, w)| &\leq c(1 + |u|^{s-1} + |v|^{s-1}), \\ |G_v(x, u, v, w)| &\leq c(1 + |u|^{s-1} + |v|^{s-1}), \end{aligned}$$

where $s \in (1, 2_\alpha^*)$ for $n \geq 3$ and $2_\alpha^* = \frac{2n}{n-\alpha}$ moreover $x \in \Omega$ a.e., $u \in \mathbb{R}$, $v \in \mathbb{R}$ and $w \in M$; if $p \in [1, \infty)$, there exists $c > 0$ such that

$$\begin{aligned} |G(x, u, v, w)| &\leq c(1 + |u|^s + |v|^s + |w|^p), \\ |G_u(x, u, v, w)| &\leq c \left(1 + |u|^{s-1} + |v|^{s-1} + |w|^{p-\frac{p}{s}}\right), \\ |G_v(x, u, v, w)| &\leq c \left(1 + |u|^{s-1} + |v|^{s-1} + |w|^{p-\frac{p}{s}}\right), \end{aligned}$$

where $s \in (1, 2_\alpha^*)$ for $n \geq 3$ and a.e. $x \in \Omega$, $u \in \mathbb{R}$, $v \in \mathbb{R}$ and $w \in \mathbb{R}^m$;

(A3) for any $u \in H_0^{\alpha/2}(\Omega)$, there exist a constant $b \in \mathbb{R}$ and some functions $\beta_1 \in L^2(\Omega)$, $\gamma_1 \in L^1(\Omega)$, such that

$$G(x, u(x), v, w) \geq -b|v|^2 - \beta_1(x)v - \gamma_1(x)$$

for any $v \in \mathbb{R}$, $w \in M$ and a.e. $x \in \Omega$, where $\rho_1^{\alpha/2} > 2b$ and ρ_1 is the principal eigenvalue of the Laplace operator with the homogeneous Dirichlet boundary values;

(A4) for any $v \in H_0^{\alpha/2}(\Omega)$, there exist a constant $B \in \mathbb{R}$ and some functions $\beta_2 \in L^2(\Omega)$, $\gamma_2 \in L^1(\Omega)$, such that

$$G(x, u, v(x), w) \leq B|u|^2 + \beta_2(x)u + \gamma_2(x)$$

for any $u \in \mathbb{R}$, $w \in M$ and $x \in \Omega$ a.e., where $\rho_1^{\alpha/2} > 2B$ and ρ_1 is the principal eigenvalue of the Laplace operator with the homogeneous Dirichlet boundary values;

(A5) for any $w \in \mathcal{W}$, the functional F_w is concave with respect to u for any $v \in H_0^{\alpha/2}(\Omega)$ and convex with respect to v for any $u \in H_0^{\alpha/2}(\Omega)$; shortly, for any $w \in \mathcal{W}$, the functional F_w is concave-convex, where F_w is defined in (7).

Remark 2.3 Under assumptions (A1) and (A2), for any $w \in \mathcal{W}$, functional F_w defined in (7) is well-defined and of C^1 -class with respect to u and v , cf. [40, Theorems C.1 and C.2].

Remark 2.4 Directly from assumptions (A1), (A2), (A3), (A4) and [40, Theorem 1.6], it follows that, for any $w \in \mathcal{W}$, the functional F_w is weakly lower semicontinuous with respect to v for any $u \in H_0^{\alpha/2}(\Omega)$ and weakly upper semicontinuous with respect to u for any $v \in H_0^{\alpha/2}(\Omega)$.

3 Existence of saddle points

In this section we shall focus our attention on study of the variational formulation of problem associated with fractional differential system (6). We shall prove that for any $w \in \mathcal{W}$, there exists a saddle point of the function of action defined in (7). Moreover, we shall demonstrate that the set of all saddle points is bounded. In doing this we will also benefit from having the following notation. For any $w \in \mathcal{W}$ denote by S_w the set all saddle point of F_w , i.e.

$$S_w = \left\{ (u_w, v_w) \in \mathbb{H}_0^{\alpha/2} : F_w(u, v_w) \leq F_w(u_w, v_w) \leq F_w(u_w, v) \right\}.$$

To prove that F_w possesses the saddle point we shall apply the following Ky Fan's Theorem.

Theorem 3.1 ([34, Theorem 5.2.2]) *Let X, Y be any linear topological spaces, $A \subset X$, $B \subset Y$ be some convex sets. Let $F : A \times B \rightarrow \mathbb{R}$ be any function satisfying conditions:*

- (a) *for any $x \in A$ the function $F(x, \cdot)$ is convex and lower semicontinuous,*
- (b) *for any $y \in B$ the function $F(\cdot, y)$ is concave and upper semicontinuous,*
- (c) *there exist $y_0 \in B$ and λ such that $\inf_{y \in B} \sup_{x \in A} F(x, y) > \lambda$ and the set $\{x \in A : F(x, y_0) \geq \lambda\}$ is compact,*
then $\sup_{x \in A} \inf_{y \in B} F(x, y) = \inf_{y \in B} \sup_{x \in A} F(x, y)$.

Now we provide the statement of the theorem on the following properties of the set of saddle points: nonemptiness and boundedness.

Theorem 3.2 (On the existence of saddle points) *If conditions (A1) – (A5) are satisfied, then for any $w \in \mathcal{W}$, there exists at least one saddle point $(u_w, v_w) \in \mathbb{H}_0^{\alpha/2}$ for the functional F_w defined in (7), and moreover there are some balls $B_1(0, r_1) \subset H_0^{\alpha/2}(\Omega)$ and $B_2(0, r_2) \subset H_0^{\alpha/2}(\Omega)$ such that, for all $w \in \mathcal{W}$, $S_w \subset B_1(0, r_1) \times B_2(0, r_2) \subset \mathbb{H}_0^{\alpha/2}$.*

If the functional F_w is additionally assumed to be strictly concave - strictly convex, then the saddle point is unique.

Proof. Let $w \in \mathcal{W}$ be fixed. First note that the functional $F_w(u, \cdot)$ is coercive for any $u \in H_0^{\alpha/2}(\Omega)$. From assumption (A3), for any $u \in H_0^{\alpha/2}(\Omega)$, there exist a constant b and functions $\beta_1 \in L^2(\Omega)$, $\gamma_1 \in L^1(\Omega)$ such that

$$F_w(u, v) \geq \int_{\Omega} \left(\frac{1}{2} \left| (-\Delta)^{\alpha/4} v(x) \right|^2 - b |v(x)|^2 - \beta_1(x) v(x) - \gamma_1(x) \right) dx.$$

The application of the fractional Poincaré inequality (5) and the Schwartz inequality lead to the following estimate:

$$F_w(u, v) \geq \left(\frac{1}{2} - b\rho_1^{-\alpha/2} \right) \|v\|_{H_0^{\alpha/2}}^2 - C_1 \|v\|_{H_0^{\alpha/2}} - C_2$$

where C_1, C_2 are some nonnegative constants. Since $\frac{1}{2} - b\rho_1^{-\alpha/2} > 0$, the functional $F_w(u, \cdot)$ is coercive. As a result, for any $u \in H_0^{\alpha/2}(\Omega)$, the functional $F_w(u, \cdot)$ attains its minimum if we also use the property of the weak lower semicontinuity of this functional. Subsequently, for any $u \in H_0^{\alpha/2}(\Omega)$, denote

$$F_w^-(u) = \min_v F_w(u, v).$$

Furthermore, from (A4) we obtain

$$F_w^-(u) \leq F_w(u, 0) \leq \int_{\Omega} \left(-\frac{1}{2} \left| (-\Delta)^{\alpha/4} u(x) \right|^2 + B |u(x)|^2 + \beta_2(x) u(x) + \gamma_2(x) \right) dx$$

for some constant B and functions $\beta_2 \in L^2(\Omega)$, $\gamma_2 \in L^1(\Omega)$.

Using the fractional Poincaré inequality (5) and the Schwartz inequality, one can get the following estimate

$$(8) \quad F_w^-(u) \leq \left(-\frac{1}{2} + B\rho_1^{-\alpha/2} \right) \|u\|_{H_0^{\alpha/2}}^2 + D_1 \|u\|_{H_0^{\alpha/2}} + D_2 = p(u)$$

where $D_1, D_2 \geq 0$. It is easily seen that the functional F_w^- is weakly upper semicontinuous. Indeed, let u_k tend to u_0 weakly in $H_0^{\alpha/2}(\Omega)$, and let $\{v_k\}_{k \in \mathbb{N}_0}$ be such that $F_w^-(u_k) = F_w(u_k, v_k) = \min_v F_w(u_k, v)$ for $k \in \mathbb{N}_0$ as we have proved such a sequence $\{v_k\}_{k \in \mathbb{N}_0}$ exists, then

$$\limsup_{k \rightarrow \infty} F_w^-(u_k) = \limsup_{k \rightarrow \infty} F_w(u_k, v_k) \leq \limsup_{k \rightarrow \infty} F_w(u_k, v_0) \leq F_w(u_0, v_0) = F_w^-(u_0).$$

Since $-\frac{1}{2} + B\rho_1^{-\alpha/2} < 0$, then for any $w \in \mathcal{W}$, the functional F_w^- attains its maximum at some point $u_w \in H_0^{\alpha/2}(\Omega)$. For any point u_w such that

$$(9) \quad F_w^-(u_w) = \max_u F_w^-(u),$$

from (A3) we obtain

$$\begin{aligned} F_w^-(u_w) &\geq F_w^-(0) = \min_v F_w(0, v) \\ &\geq \min_v \left(\left(\frac{1}{2} - b\rho_1^{-\alpha/2} \right) \|v\|_{H_0^{\alpha/2}}^2 - C_1 \|v\|_{H_0^{\alpha/2}} - C_2 \right) = \eta > -\infty \end{aligned}$$

where b, C_1, C_2, η are some constants and $\frac{1}{2} - b\rho_1^{-\alpha/2} > 0$. Note that η does not depend on control w . Moreover, it is important to notice that, for any maximizer u_w satisfying (9), there exists $r_1 > 0$ such that for any $w \in \mathcal{W}$

$$(10) \quad u_w \in \{u : F_w^-(u) \geq \eta\} \subset \{u : p(u) \geq \eta\} \subset B_1(0, r_1)$$

where p is defined in (8). We have thus checked that, for any $w \in \mathcal{W}$, there exists at least one u_w such that

$$F_w^-(u_w) = \max_u F_w^-(u) = \max_u \left[\min_v F_w(u, v) \right].$$

In a similar way one can demonstrate that, for any $w \in \mathcal{W}$, there exists at least one $v_w \in H_0^{\alpha/2}(\Omega)$ such that

$$(11) \quad F_w^+(v_w) = \min_v F_w^+(v) = \min_v \left[\max_u F_w(u, v) \right]$$

where $F_w^+(v) = \max_u F_w(u, v)$ and moreover, there is $r_2 > 0$ such that

$$(12) \quad v_w \in B_2(0, r_2)$$

for any v_w satisfying (11).

Furthermore, since the function $v \rightarrow \max_u F_w(u, v)$ attains its minimum, therefore there is a number λ such that

$$\lambda < \min_v \max_u F_w(u, v) \leq \max_u F_w(u, 0).$$

and

$$\left\{ u \in H_0^{\alpha/2}(\Omega) : F_w(u, 0) \geq \lambda \right\} \subset \left\{ u \in H_0^{\alpha/2}(\Omega) : p(u) \geq \lambda \right\} = A_0$$

where p is defined in (8). Moreover, since A_0 is relatively compact in the weak topology of $H_0^{\alpha/2}(\Omega)$ as it is a bounded subset of the reflexive space, it follows that the set $\left\{ u \in H_0^{\alpha/2}(\Omega) : F_w(u, 0) \geq \lambda \right\}$ is weakly compact. Additionally, by (A5), F_w is concave-convex for any $w \in \mathcal{W}$. In that way we have demonstrated that all assertions of Ky Fan's Theorem are satisfied. Therefore, $\max_u \min_v F_w(u, v) = \min_v \max_u F_w(u, v)$ for any $w \in \mathcal{W}$. Subsequently, for any $v \in H_0^{\alpha/2}(\Omega)$, we have

$$\begin{aligned} F_w(u_w, v_w) &\leq \max_u F_w(u, v_w) = F_w^+(v_w) = \min_v F_w^+(v) = \min_v \left[\max_u F_w(u, v) \right] = \max_u \left[\min_v F_w(u, v) \right] \\ &= \max_u F_w^-(u) = F_w^-(u_w) = \min_v F_w(u_w, v) \leq F_w(u_w, v). \end{aligned}$$

In a similar way one can verify that for any $u \in H_0^{\alpha/2}(\Omega)$

$$F_w(u_w, v_w) \geq F_w(u, v_w).$$

Hence, for any $u \in H_0^{\alpha/2}(\Omega)$ and $v \in H_0^{\alpha/2}(\Omega)$, the following inequalities

$$F_w(u, v_w) \leq F_w(u_w, v_w) \leq F_w(u_w, v)$$

hold. Therefore, for any $w \in \mathcal{W}$, there exists at least one saddle point of the functional F_w and moreover by (10) and (12), $S_w \subset B_1(0, r_1) \times B_2(0, r_2)$. This finishes the proof. ■

Remark 3.1 *It is well-known that the set of weak solutions of (6) coincides with the set of saddle points of the functional of action defined in (7) if the functional of action is concave-convex. Furthermore, problem (6) has a unique solution if F_w is strictly concave-strictly convex.*

4 Continuous dependence

A natural question to ask is how (u, v) varies as w changes. Now we look for conditions under which solutions of the variational problem are stable. By stability here we understand the continuous dependence of saddle points on controls. In order to state these conditions succinctly, we introduce some notation and terminology. Let $\{w_k\}_{k \in \mathbb{N}_0}$ be an arbitrary sequence of elements from \mathcal{W} . Next, by $\{\varphi_k\}_{k \in \mathbb{N}_0}$ we denote a sequence of functionals of action such that

$$(13) \quad \varphi_k(u, v) = F_{w_k}(u, v), \quad k \in \mathbb{N}_0,$$

where F_w is defined in (7) and by S_k the set of saddle points of the functional φ_k for $k \in \mathbb{N}_0$, i.e.

$$(14) \quad S_k = \left\{ (\bar{u}, \bar{v}) \in \mathbb{H}_0^{\alpha/2} : \varphi_k(\bar{u}, \bar{v}) = \max_u \min_v \varphi_k(u, v) = \min_v \max_u \varphi_k(u, v) \right\}.$$

In view of Theorem 3.2, for any k , there exists at least one saddle point of the functional φ_k , so that the set S_k is nonempty and there exist $r_1, r_2 > 0$ such that $S_k \subset B_1(0, r_1) \times B_2(0, r_2) \subset \mathbb{H}_0^{\alpha/2}$.

Before we prove the next theorem, we recall the definition of the upper Kuratowski-Painlevé limit of the sets X_k in the topological space (\mathcal{H}, τ) , where $\{X_k\}_{k \in \mathbb{N}}$ is a sequence of subsets of the space \mathcal{H} with topology τ , cf. [2].

Definition 4.1 *The upper limit of the sequence $\{X_k\}_{k \in \mathbb{N}}$ is defined as the set of all cluster points of sequences $\{x_k\}_{k \in \mathbb{N}}$ such that $x_k \in X_k$ for $k \in \mathbb{N}$. The upper limit of $\{X_k\}_{k \in \mathbb{N}}$ in (\mathcal{H}, τ) will be denoted by $(\tau) \text{Lim sup } X_k$.*

Additionally, X_k is said to tend to X_0 in (\mathcal{H}, τ) if $(\tau) \text{Lim sup } X_k \subset X_0$. In this paper $\mathcal{H} = \mathbb{H}_0^{\alpha/2}$ considered with the weak topology denoted by (w) or the strong topology denoted by (s) , $X_k = S_k$ where S_k is defined in (14) and $x_k = (u_k, v_k)$ where (u_k, v_k) is a saddle point of φ_k defined in (13).

4.1 Strong convergence of controls

Proposition 4.2 *If conditions (A1)–(A5) are satisfied and a sequence of controls w_k tends to w_0 in $L^p(\Omega, \mathbb{R}^m)$, then $(w) \text{Lim sup } S_k \neq \emptyset$ and $(w) \text{Lim sup } S_k \subset S_0$ in $\mathbb{H}_0^{\alpha/2}$ where S_k are given by (14).*

Proof. We begin by proving that φ_k converges uniformly to φ_0 on $B_1(0, r_1) \times B_2(0, r_2)$ where $B_1(0, r_1), B_2(0, r_2)$ are balls from Theorem 3.2 such that the set of all saddle points of φ_k denoted by S_k is contained in $B_1(0, r_1) \times B_2(0, r_2)$. To do this let $v \in H_0^{\alpha/2}(\Omega)$ be an arbitrary point and suppose that, on the contrary, the sequence $\{\varphi_k(\cdot, v)\}_{k \in \mathbb{N}}$ does not converge to $\varphi_0(\cdot, v)$ uniformly on $B_1(0, r_1)$. This means that there exists a sequence $\{u_l\} \subset B_1(0, r_1)$ and a positive constant ε such that

$$|\varphi_k(u_l, v) - \varphi_0(u_l, v)| \geq \varepsilon \text{ for } k \in \mathbb{N}.$$

Passing to a subsequence if necessary, one can assume that $u_l \rightharpoonup u_0 \in B_1(0, r_1)$ weakly in $H_0^{\alpha/2}(\Omega)$. It is a simple observation, using the triangle inequality, that

$$\begin{aligned} |\varphi_k(u_l, v) - \varphi_0(u_l, v)| &\leq \int_{\Omega} |G(x, u_l(x), v(x), w_k(x)) - G(x, u_l(x), v(x), w_0(x))| dx \\ &\leq \int_{\Omega} |G(x, u_l(x), v(x), w_k(x)) - G(x, u_0(x), v(x), w_0(x))| dx \\ &\quad + \int_{\Omega} |G(x, u_l(x), v(x), w_0(x)) - G(x, u_0(x), v(x), w_0(x))| dx \end{aligned}$$

for $k \in \mathbb{N}$. The lower estimate by ε leads to the contradiction with the upper bound as all the above integrals tend to zero. To observe this it is enough to apply Krasnoselskii Theorem [28, Theorem 2] on the continuity of the superposition operator of the operators

$$\begin{aligned} L^s(\Omega) \times L^p(\Omega, \mathbb{R}^m) \ni (u, w) &\mapsto G(\cdot, u(\cdot), v(\cdot), w(\cdot)) \in L^1(\Omega) \\ L^s(\Omega) \ni u &\mapsto G(\cdot, u(\cdot), v(\cdot), w(\cdot)) \in L^1(\Omega) \end{aligned}$$

since (A2) holds. Next, apply the same arguments to get the uniform convergence of the sequence $\{\varphi_k(u, \cdot)\}_{k \in \mathbb{N}}$ on a ball $B_2(0, r_2)$. Therefore, $\varphi_k \rightrightarrows \varphi_0$ on $B_1(0, r_1) \times B_2(0, r_2)$.

Let us denote

$$m_k = \max_u \min_v \varphi_k(u, v) = \max_{u \in B_1(0, r_1)} \min_{v \in B_2(0, r_2)} \varphi_k(u, v) \text{ for } k \in \mathbb{N}_0.$$

Since $\varphi_k \rightrightarrows \varphi_0$ on $B_1(0, r_1) \times B_2(0, r_2)$, for any $\varepsilon > 0$, there exists K_0 such that

$$\varphi_k(u, v) \leq \varphi_0(u, v) + \varepsilon$$

for any $(u, v) \in B_1(0, r_1) \times B_2(0, r_2)$ and $k > K_0$. This implies that

$$\min_{v \in B_2(0, r_2)} \varphi_k(u, v) \leq \min_{v \in B_2(0, r_2)} \varphi_0(u, v) + \varepsilon$$

for any $u \in B_1(0, r_1)$ and $k > K_0$. Consequently,

$$\max_{u \in B_1(0, r_1)} \min_{v \in B_2(0, r_2)} \varphi_k(u, v) \leq \max_{u \in B_1(0, r_1)} \min_{v \in B_2(0, r_2)} \varphi_0(u, v) + \varepsilon$$

for $k > K_0$. Thus $m_k - m_0 \leq \varepsilon$ for sufficiently large k . In a similar way it is possible to show that $-\varepsilon \leq m_k - m_0$ for sufficiently large k . In this way we have proved that m_k tends to m_0 as $k \rightarrow \infty$.

Next, let $\{(u_k, v_k)\}_{k \in \mathbb{N}}$ be an arbitrary sequence of saddle points, such that $(u_k, v_k) \in S_k$ for $k \in \mathbb{N}$. From Theorem 3.2, for any $k \in \mathbb{N}$, the set S_k is nonempty and there exist $r_1 > 0$ and $r_2 > 0$ such that $S_k \subset B_1(0, r_1) \times B_2(0, r_2)$ for every k , i.e. the sequence $\{(u_k, v_k)\}_{k \in \mathbb{N}}$ is bounded. Moreover, the space $\mathbb{H}_0^{\alpha/2}$ is reflexive, which implies that the sequence $\{(u_k, v_k)\}_{k \in \mathbb{N}}$ is weakly compact, therefore the set of its cluster points with respect of weak topology of $\mathbb{H}_0^{\alpha/2}$ is nonempty. This means that $(w) \limsup S_k \neq \emptyset$.

Let $(u_0, v_0) \in B_1(0, r_1) \times B_2(0, r_2)$ be any cluster point of the sequence $\{(u_k, v_k)\}_{k \in \mathbb{N}}$. Going, if necessary, to a subsequence, we may assume that $\{(u_k, v_k)\}_{k \in \mathbb{N}}$ tends to (u_0, v_0) weakly in $\mathbb{H}_0^{\alpha/2}$. We shall show that $(u_0, v_0) \in S_0$. Suppose on the contrary that (u_0, v_0) does not belong to S_0 . Let (\tilde{u}, \tilde{v}) be an element of S_0 . So, we have $\varphi_0(u_0, v_0) \neq \varphi_0(\tilde{u}, \tilde{v})$. First, consider the case when $\varphi_0(\tilde{u}, \tilde{v}) - \varphi_0(u_0, v_0) = \lambda < 0$. In that case we have

$$\begin{aligned} m_k - m_0 &= \varphi_k(u_k, v_k) - \varphi_0(u_0, v_0) \leq \varphi_k(u_k, \tilde{v}) - \varphi_0(u_0, v_0) \\ &= (\varphi_k(u_k, \tilde{v}) - \varphi_0(u_k, \tilde{v})) + (\varphi_0(u_k, \tilde{v}) - \varphi_0(\tilde{u}, \tilde{v})) \\ &\quad + (\varphi_0(\tilde{u}, \tilde{v}) - \varphi_0(u_0, v_0)). \end{aligned}$$

From uniform convergence of φ_k to φ_0 on $B_1(0, r_1) \times B_2(0, r_2)$ and the weak upper semicontinuity of $\varphi_0(\cdot, v)$ we have

$$\begin{aligned} \lim_{k \rightarrow \infty} [\varphi_k(u_k, \tilde{v}) - \varphi_0(u_k, \tilde{v})] &= 0, \\ \limsup_{k \rightarrow \infty} [\varphi_0(u_k, \tilde{v}) - \varphi_0(\tilde{u}, \tilde{v})] &\leq 0. \end{aligned}$$

This implies that $\limsup_{k \rightarrow \infty} (m_k - m_0) \leq \lambda < 0$. We have thus got a contradiction with the previously proved fact that $m_k \rightarrow m_0$ as $k \rightarrow \infty$. Similarly, we obtain a contradiction in the case when $\lambda > 0$. Therefore, $(u_0, v_0) \in S_0$ and $(w) \limsup S_k \subset S_0$ in $\mathbb{H}_0^{\alpha/2}$, which concludes the proof. ■

Proposition 4.3 *If conditions (A1) – (A5) are satisfied and w_k tends to w_0 in $L^p(\Omega, \mathbb{R}^m)$, then $(s) \limsup S_k \neq \emptyset$ and $(s) \limsup S_k \subset S_0$ in $\mathbb{H}_0^{\alpha/2}$.*

Proof. We start with a proof of the uniform convergence of φ'_k to φ'_0 on $B_1(0, r_1) \times B_2(0, r_2)$ where as before $B_1(0, r_1)$, $B_2(0, r_2)$ are balls from Theorem 3.2 such that for all $w \in \mathcal{W}$, the set of all saddle points of φ_k denoted by S_k is a subset of $B_1(0, r_1) \times B_2(0, r_2)$.

Let $v \in H_0^{\alpha/2}(\Omega)$ be an arbitrary point. First, suppose that the sequence $\left\{ \frac{\partial \varphi_k}{\partial u}(\cdot, v) \right\}_{k \in \mathbb{N}}$ does not converge to $\frac{\partial \varphi_0}{\partial u}(\cdot, v)$ uniformly on $B_1(0, r_1)$. This means that there exists a sequence $\{u_l\} \subset B_1(0, r_1)$ and a positive constant ε such that

$$\left| \left\langle \frac{\partial \varphi_k}{\partial u}(u_l, v) - \frac{\partial \varphi_0}{\partial u}(u_l, v), g_l \right\rangle \right| \geq \varepsilon \text{ for } k \in \mathbb{N}$$

and $\{g_l\} \subset B_1(0, r_1)$. Passing to a subsequence if necessary, assume that $u_l \rightharpoonup u_0 \in B_1(0, r_1)$. Clearly,

$$\begin{aligned} \left| \left\langle \frac{\partial \varphi_k}{\partial u}(u_l, v) - \frac{\partial \varphi_0}{\partial u}(u_l, v), g_l \right\rangle \right| &\leq \int_{\Omega} |(G_u(x, u_l(x), v(x), w_k(x)) - G_u(x, u_l(x), v(x), w_0(x))) g_l(x)| dx \\ &\leq \int_{\Omega} |G_u(x, u_l(x), v(x), w_k(x)) - G_u(x, u_0(x), v(x), w_0(x))| |g_l(x)| dx \\ &\quad + \int_{\Omega} |(G_u(x, u_l(x), v(x), w_0(x)) - G_u(x, u_0(x), v(x), w_0(x)))| |g_l(x)| dx \end{aligned}$$

for $k \in \mathbb{N}$. The above integrals tend to zero. This is an immediate consequence of Krasnoselskii Theorem [28, Theorem 2] on the continuity of the superposition operator of the operators

$$\begin{aligned} L^s(\Omega) \times L^p(\Omega, \mathbb{R}^m) \ni (u, w) &\mapsto G_u(\cdot, u(\cdot), v(\cdot), w(\cdot)) \in L^{\frac{s}{s-1}}(\Omega) \\ L^s(\Omega) \ni u &\mapsto G_u(\cdot, u(\cdot), v(\cdot), w(\cdot)) \in L^{\frac{s}{s-1}}(\Omega) \end{aligned}$$

by (A2) and using the fact that the sequence $\{g_l\}$ is bounded. Next, in similar fashion, the uniform convergence of the sequence $\left\{ \frac{\partial \varphi_k}{\partial u}(u, \cdot) \right\}_{k \in \mathbb{N}}$ on a ball $B_2(0, r_2)$ can be easily verified. As a result, $\varphi'_k \rightrightarrows \varphi'_0$ on $B_1(0, r_1) \times B_2(0, r_2)$.

Let $\{(u_k, v_k)\} \subset \mathbb{H}_0^{\alpha/2}$ be a sequence such that $(u_k, v_k) \in S_k$ for $k \in \mathbb{N}$. Since, for any $k \in \mathbb{N}$, $S_k \subset B_1(0, r_1) \times B_2(0, r_2)$, for some $r_1, r_2 > 0$ (cf. Theorem 3.2), we may assume, without loss of generality, that (u_k, v_k) converges weakly to some $(u_0, v_0) \in B_1(0, r_1) \times B_2(0, r_2)$ in $\mathbb{H}_0^{\alpha/2}$. Our aim is now to show that $(u_k, v_k) \rightarrow (u_0, v_0)$ strongly in $\mathbb{H}_0^{\alpha/2}$. Actually, by direct calculations we get

$$\begin{aligned} &\langle \varphi'_0(u_k, v_k) - \varphi'_0(u_0, v_0), (u_0 - u_k, v_k - v_0) \rangle \\ &= \|u_k - u_0\|_{H_0^{\alpha/2}}^2 + \|v_k - v_0\|_{H_0^{\alpha/2}}^2 \\ &\quad + \int_{\Omega} (G_u(x, u_k(x), v_k(x), w_0(x)) - G_u(x, u_0(x), v_0(x), w_0(x))) (u_0(x) - u_k(x)) dx \\ &\quad + \int_{\Omega} (G_v(x, u_k(x), v_k(x), w_0(x)) - G_v(x, u_0(x), v_0(x), w_0(x))) (v_k(x) - v_0(x)) dx. \end{aligned}$$

Since $\varphi'_k \rightrightarrows \varphi'_0$ on $B_1(0, r_1) \times B_2(0, r_2)$, $\varphi'_0(u_k, v_k) \rightarrow 0$ and therefore the left side of the above equality tends to 0. We shall show that the last two integrals above tend to zero. The condition (A2) and the Hölder inequality lead to the estimates:

$$\begin{aligned} &\left| \int_{\Omega} (G_u(x, u_k(x), v_k(x), w_0(x)) - G_u(x, u_0(x), v_0(x), w_0(x))) (u_0(x) - u_k(x)) dx \right| \\ &\leq \left(\int_{\Omega} |G_u(x, u_k(x), v_k(x), w_0(x)) - G_u(x, u_0(x), v_0(x), w_0(x))|^{\frac{s}{s-1}} dx \right)^{\frac{s-1}{s}} \left(\int_{\Omega} |u_0(x) - u_k(x)|^s dx \right)^{\frac{1}{s}} \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{\Omega} (G_v(x, u_k(x), v_k(x), w_0(x)) - G_v(x, u_0(x), v_0(x), w_0(x))) (v_k(x) - v_0(x)) dx \right| \\ & \leq \left(\int_{\Omega} |G_v(x, u_k(x), v_k(x), w_0(x)) - G_v(x, u_0(x), v_0(x), w_0(x))|^{\frac{s}{s-1}} dx \right)^{\frac{s-1}{s}} \left(\int_{\Omega} |v_k(x) - v_0(x)|^s dx \right)^{\frac{1}{s}}. \end{aligned}$$

Since $H_0^{\alpha/2}(\Omega)$ is compactly embedded into $L^s(\Omega)$ for $s \in (1, 2_{\alpha}^*)$ if $n > 2$ and since both first integrals in the above estimates are bounded, it follows that $(u_k, v_k) \rightarrow (u_0, v_0) \in S_0$ in the strong topology of $\mathbb{H}_0^{\alpha/2}$, i.e. $(s) \limsup S_k \neq \emptyset$. Obviously, $(s) \limsup S_k \subset S_0$ in $\mathbb{H}_0^{\alpha/2}$, which is a direct consequence of $(w) \limsup S_k \subset S_0$ in $\mathbb{H}_0^{\alpha/2}$ as proved in Proposition 4.4 and the inclusion $(s) \limsup S_k \subset (w) \limsup S_k$. This concludes the proof. \blacksquare

Remark 4.1 In other words, from Proposition 4.3 it follows that the set-valued mapping

$$L^p(\Omega, \mathbb{R}^m) \ni w_k \mapsto S_k \subset \mathbb{H}_0^{\alpha/2}$$

is well-defined and upper semicontinuous with respect to the strong topology of $L^p(\Omega, \mathbb{R}^m)$ and the strong topology of $\mathbb{H}_0^{\alpha/2}$. If additionally each S_k is a singleton, i.e., $S_k = \{(u_k, v_k)\}$ then $(u_k, v_k) \rightarrow (u_0, v_0)$ provided $w_k \rightarrow w_0$ in $L^p(\Omega, \mathbb{R}^m)$.

4.2 Weak convergence of controls

To achieve stronger results which are useful in optimization theory, it is necessary to weaken the notion of the convergence of controls. As a side effect we should therefore narrow the class of equations under considerations. Namely, in this section, we shall assume that the integrand G is linear with respect to control w , i.e. the function G will take the form

$$(15) \quad G(x, u, v, w) = G^1(x, u, v) + G^2(x, u, v) w$$

where $G^1 : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $G^2 : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}^m$, $w \in \mathbb{R}^m$.

Obviously, in this case the boundary value problem (6) takes the form

$$(16) \quad \begin{cases} -(-\Delta)^{\alpha/2} u(x) + G_u^1(x, u(x), v(x)) + G_u^2(x, u(x), v(x)) w(x) = 0 & \text{in } \Omega \\ (-\Delta)^{\alpha/2} v(x) + G_v^1(x, u(x), v(x)) + G_v^2(x, u(x), v(x)) w(x) = 0 & \text{in } \Omega \\ u(x) = 0, v(x) = 0 & \text{on } \partial\Omega \end{cases}$$

and the functional of action now assumes the form

$$F_w(u, v) = \int_{\Omega} \left(\frac{1}{2} \left| (-\Delta)^{\alpha/4} v(x) \right|^2 - \frac{1}{2} \left| (-\Delta)^{\alpha/4} u(x) \right|^2 + G^1(x, u(x), v(x)) + G^2(x, u(x), v(x)) w(x) \right) dx$$

where $u \in H_0^{\alpha/2}(\Omega)$, $v \in H_0^{\alpha/2}(\Omega)$, $w \in L^p(\Omega, \mathbb{R}^m)$ with $p > 1$.

From now on we impose the following conditions on G^1, G^2 :

(A1') the functions $G^1, G_u^1, G_v^1, G^2, G_u^2, G_v^2$ are measurable with respect to x for any $(u, v) \in \mathbb{R}^2$ and continuous with respect to (u, v) for a.e. $x \in \Omega$;

(A2') for $p \in (1, \infty)$, there exists a constant $c > 0$ such that

$$\begin{aligned} |G_u^1(x, u, v)| &\leq c \left(1 + |u|^{s-1} + |v|^{s-1} \right) \\ |G_v^1(x, u, v)| &\leq c \left(1 + |u|^{s-1} + |v|^{s-1} \right) \\ |G_u^2(x, u, v)| &\leq c \left(1 + |u|^{s-1-\frac{s}{p}} + |u|^{s-1-\frac{s}{p}} \right) \\ |G_v^2(x, u, v)| &\leq c \left(1 + |u|^{s-1-\frac{s}{p}} + |u|^{s-1-\frac{s}{p}} \right) \end{aligned}$$

for $x \in \Omega$ a.e., $u \in \mathbb{R}$, $v \in \mathbb{R}$ and $s \in \left(1 + \frac{1}{p-1}, 2_\alpha^*\right)$ where $2_\alpha^* = \frac{2n}{n-\alpha} > 2$ and $p > \frac{2n}{n+\alpha}$;
for $p = \infty$, there exist a constant $c > 0$ such that

$$\begin{aligned} |G_u^1(x, u, v)| &\leq c \left(1 + |u|^{s-1} + |v|^{s-1}\right) \\ |G_v^1(x, u, v)| &\leq c \left(1 + |u|^{s-1} + |v|^{s-1}\right) \\ |G_u^2(x, u, v)| &\leq c \left(1 + |u|^{s-1} + |v|^{s-1}\right) \\ |G_v^2(x, u, v)| &\leq c \left(1 + |u|^{s-1} + |v|^{s-1}\right) \end{aligned}$$

for $x \in \Omega$ a.e., $u \in \mathbb{R}$, $v \in \mathbb{R}$ and $s \in (1, 2_\alpha^*)$.

Obviously, assumptions $(A1')$, $(A2')$ imply that the function G satisfies $(A1)$ and $(A2)$. Moreover, we shall suppose that the function G given by (15) meets conditions $(A3)$, $(A4)$, $(A5)$. For this more specific form of the problem, the claim of the theorem on the existence and the continuous dependence can be strengthened. To draw the same conclusion this time, it suffices to assume only the weak convergence of controls.

Let $\{w_k\}_{k \in \mathbb{N}}$ be some sequence of controls. We shall prove the following proposition.

Proposition 4.4 *Suppose that the function G is of the form (15) and satisfies conditions $(A1')$, $(A2')$, $(A3)$, $(A4)$, $(A5)$. Moreover, the sequence of controls w_k converges to w_0 in the weak topology of $L^p(\Omega, \mathbb{R}^m)$ for $p \in \left(\frac{2n}{n+\alpha}, \infty\right)$. Then $(s) \text{Lim sup } S_k \neq \emptyset$ and $(s) \text{Lim sup } S_k \subset S_0$ in $\mathbb{H}_0^{\alpha/2}$.*

Proof. The proof is similar in spirit to that of Propositions 4.2 and 4.3. Although this proof runs along similar lines, there is need of some subtle adjustments required to fit the arguments to new framework. In fact, to prove that $(w) \text{Lim sup } S_k \neq \emptyset$ and $(w) \text{Lim sup } S_k \subset S_0$ in $\mathbb{H}_0^{\alpha/2}$ we proceed along the same lines as in the proof of Proposition 4.2. The only thing to check now is the uniform convergence of φ_k to φ_0 on $B_1(0, r_1) \times B_2(0, r_2)$. Let $v \in H_0^{\alpha/2}(\Omega)$ be an arbitrary point. Suppose, to derive a contradiction, that the sequence $\{\varphi_k(\cdot, v)\}_{k \in \mathbb{N}}$ does not converge to $\varphi_0(\cdot, v)$ uniformly on $B_1(0, r_1)$. This means that there exist a sequence $\{u_l\} \subset B_1(0, r_1)$ and a positive constant ε such that

$$(17) \quad |\varphi_k(u_l, v) - \varphi_0(u_l, v)| \geq \varepsilon \text{ for } k \in \mathbb{N}.$$

Passing, if necessary, to a subsequence let us assume that $u_l \rightharpoonup u_0 \in B_1(0, r_1)$. By direct calculations, we get

$$\begin{aligned} |\varphi_k(u_l, v) - \varphi_0(u_l, v)| &\leq \int_{\Omega} |G^2(x, u_l(x), v(x)) (w_k(x) - w_0(x))| dx \\ &\leq \int_{\Omega} |(G^2(x, u_l(x), v(x)) - G^2(x, u_0(x), v(x))) (w_k(x) - w_0(x))| dx \\ &\quad + \int_{\Omega} |G^2(x, u_0(x), v(x)) (w_k(x) - w_0(x))| dx \\ &\leq \left(\int_{\Omega} |G^2(x, u_l(x), v(x)) - G^2(x, u_0(x), v(x))|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_{\Omega} |w_k(x) - w_0(x)|^p dx \right)^{\frac{1}{p}} \\ &\quad + \int_{\Omega} |G^2(x, u_0(x), v(x)) (w_k(x) - w_0(x))| dx \end{aligned}$$

for $k \in \mathbb{N}$. Now we end up with a contradiction with (17) since the above integrals tend to zero. To observe this convergence one can invoke [28, Theorem 2] to get the continuity of the mapping

$$L^s(\Omega) \times L^s(\Omega) \ni (u, v) \mapsto G^2(\cdot, u(\cdot), v(\cdot)) \in L^{\frac{p}{p-1}}(\Omega)$$

and use the assumption (A2') together with the weak convergence of controls in $L^p(\Omega, \mathbb{R}^m)$. The same arguments apply to the uniform convergence of the sequence $\{\varphi_k(u, \cdot)\}_{k \in \mathbb{N}}$ on a ball $B_2(0, r_2)$. In that way one can demonstrate that $\varphi_k \rightharpoonup \varphi_0$ on $B_1(0, r_1) \times B_2(0, r_2)$.

To prove that $(s)\text{Lim sup } S_k \neq \emptyset$ and $(s)\text{Lim sup } S_k \subset S_0$ in $\mathbb{H}_0^{\alpha/2}$ we proceed in the exactly same way as in the proof of Proposition 4.3. We need to show that φ'_k converges uniformly to φ'_0 on $B_1(0, r_1) \times B_2(0, r_2)$.

Let $v \in H_0^{\alpha/2}(\Omega)$ be an arbitrary point. In a contradiction with the claim, suppose that the sequence $\left\{\frac{\partial \varphi_k}{\partial u}(\cdot, v)\right\}_{k \in \mathbb{N}}$ does not converge to $\frac{\partial \varphi_0}{\partial u}(\cdot, v)$ uniformly on $B_1(0, r_1)$. This means again that there exist a sequence $\{u_l\} \subset B_1(0, r_1)$ and a positive constant ε such that

$$\left| \left\langle \frac{\partial \varphi_k}{\partial u}(u_l, v) - \frac{\partial \varphi_0}{\partial u}(u_l, v), g_l \right\rangle \right| \geq \varepsilon \text{ for } k \in \mathbb{N}$$

and $\{g_l\} \subset B_1(0, r_1)$. Passing to a subsequence one can assume that $u_l \rightharpoonup u_0 \in B_1(0, r_1)$. It can be easily verified that

$$\begin{aligned} & \left| \left\langle \frac{\partial \varphi_k}{\partial u}(u_l, v) - \frac{\partial \varphi_0}{\partial u}(u_l, v), g_l \right\rangle \right| \\ & \leq \int_{\Omega} |(G_u^2(x, u_l(x), v(x)) w_k(x) - G_u^2(x, u_l(x), v(x))) w_0(x) g_l(x)| dx \\ & \leq \int_{\Omega} |(G_u^2(x, u_l(x), v(x)) - G_u^2(x, u_0(x), v(x))) (w_k(x) - w_0(x))| |g_l(x)| dx \\ & + \int_{\Omega} |G_u^2(x, u_0(x), v(x)) (w_k(x) - w_0(x))| |g_l(x)| dx \\ & \leq \left(\int_{\Omega} |G_u^2(x, u_l(x), v(x)) - G_u^2(x, u_0(x), v(x))|^{\frac{ps}{p(s-1)-s}} \right)^{\frac{p(s-1)-s}{ps}} \|w_k - w_0\|_{L^p} \|g_l\|_{L^s} \\ & + \int_{\Omega} |G_u^2(x, u_0(x), v(x)) (w_k(x) - w_0(x))| |g_l(x)| dx \end{aligned}$$

for $k \in \mathbb{N}$. The only thing to check is the convergence to zero of the above integrals. The assumption (A2'), boundedness of the sequences $\{g_l\}$, $\{\|w_k\|_{L^p}\}$ as well as continuity of the operators

$$\begin{aligned} L^s(\Omega) \times L^p(\Omega, \mathbb{R}^m) & \ni (u, w) \mapsto G_u^2(\cdot, u(\cdot), v(\cdot)) w(\cdot) \in L^{\frac{s}{s-1}}(\Omega) \\ L^s(\Omega) & \ni u \mapsto G_u^2(\cdot, u(\cdot), v(\cdot)) \in L^{\frac{ps}{p(s-1)-s}}(\Omega, \mathbb{R}^m) \end{aligned}$$

make it possible to draw the desired conclusion. Likewise, one can show the uniform convergence of the sequence $\left\{\frac{\partial \varphi_k}{\partial u}(u, \cdot)\right\}_{k \in \mathbb{N}}$ on a ball $B_2(0, r_2)$. Therefore, $\varphi'_k \rightharpoonup \varphi'_0$ on $B_1(0, r_1) \times B_2(0, r_2)$. The rest of the proof follows as the proofs of Propositions 4.2 and 4.3. ■

Proposition 4.5 *Assume that G is of the form (15) and satisfies conditions (A1'), (A2'), (A3), (A4), (A5). Moreover, the sequence of controls w_k tends to w_0 in the weak * topology of $L^\infty(\Omega, \mathbb{R}^m)$. Then $(s)\text{Lim sup } S_k \neq \emptyset$ and $(s)\text{Lim sup } S_k \subset S_0$ in $\mathbb{H}_0^{\alpha/2}$.*

Sketch of the proof. As it was pointed out in the proof of Proposition 4.4, all we need is to demonstrate that $\varphi_k \rightharpoonup \varphi_0$ on $B_1(0, r_1) \times B_2(0, r_2)$ and $\varphi'_k \rightharpoonup \varphi'_0$ on $B_1(0, r_1) \times B_2(0, r_2)$. Assume on the contrary. Note the following estimates (analogously we consider the sequence $\{v_l\}$ and an arbitrary u)

$$(18) \quad \begin{aligned} |\varphi_k(u_l, v) - \varphi_0(u_l, v)| & \leq \int_{\Omega} |(G^2(x, u_l(x), v(x)) - G^2(x, u_0(x), v(x))) (w_k(x) - w_0(x))| dx \\ & + \int_{\Omega} |G^2(x, u_0(x), v(x)) (w_k(x) - w_0(x))| dx \end{aligned}$$

and

$$(19) \quad \left| \left\langle \frac{\partial \varphi_k}{\partial u}(u_l, v) - \frac{\partial \varphi_0}{\partial u}(u_l, v), g_l \right\rangle \right| \leq \int_{\Omega} |G_u^2(x, u_l(x), v(x)) - G_u^2(x, u_0(x), v(x))| (w_k(x) - w_0(x)) |g_l(x)| dx \\ + \int_{\Omega} |G_u^2(x, u_0(x), v(x))| (w_k(x) - w_0(x)) |g_l(x)| dx$$

for fixed $v \in H_0^{\alpha/2}(\Omega)$ and some sequences $\{u_l\} \subset B_1(0, r_1)$, $\{g_l\} \subset B_1(0, r_1)$. Since $\{w_k\}_{k \in \mathbb{N}}$ tends to w_0 in the weak $*$ topology of $L^\infty(\Omega, \mathbb{R}^m)$ and since the operators

$$L^s(\Omega) \times L^s(\Omega) \ni (u, v) \mapsto G^2(\cdot, u(\cdot), v(\cdot)) \in L^1(\Omega), \\ L^s(\Omega) \times L^s(\Omega) \ni (u, v) \mapsto G_u^2(\cdot, u(\cdot), v(\cdot)) \in L^1(\Omega),$$

are continuous, it follows, as $k \rightarrow \infty$, that

$$\int_{\Omega} |G^2(x, u_0(x), v(x))| (w_k(x) - w_0(x)) dx \rightarrow 0, \\ \int_{\Omega} |G_u^2(x, u_0(x), v(x))| (w_k(x) - w_0(x)) dx \rightarrow 0.$$

Furthermore, due to the continuity of the following operators

$$L^s(\Omega) \ni u \mapsto G^2(\cdot, u(\cdot), v(\cdot)) \in L^1(\Omega), \\ L^s(\Omega) \ni u \mapsto G_u^2(\cdot, u(\cdot), v(\cdot)) \in L^1(\Omega),$$

and boundedness of $\{\|w_k - w_0\|_{L^\infty}\}$ we get that all right side of integrals in (18) and (19) tend to zero which contradicts our supposition. The rest of the proof follows the lines of the proofs of Propositions 4.2 and 4.3. ■

Remark 4.2 In Propositions 4.4 and 4.5, we have proved that the set-valued mapping

$$L^p(\Omega, \mathbb{R}^m) \ni w_k \mapsto S_k \subset \mathbb{H}_0^{\alpha/2}$$

is well-defined and upper semicontinuous with respect to either the weak topology of $L^p(\Omega, \mathbb{R}^m)$ for $p \in \left(\frac{2n}{n+\alpha}, \infty\right)$ or the weak $*$ topology of $L^\infty(\Omega, \mathbb{R}^m)$, and the strong topology of $\mathbb{H}_0^{\alpha/2}$. If additionally each S_k is a singleton, i.e., $S_k = \{(u_k, v_k)\}$, then $(u_k, v_k) \rightarrow (u_0, v_0)$ in $\mathbb{H}_0^{\alpha/2}$ provided either $w_k \rightharpoonup w_0$ weakly in $L^p(\Omega, \mathbb{R}^m)$ or $w_k \xrightarrow{*} w_0$ weakly $*$ in $L^\infty(\Omega, \mathbb{R}^m)$.

5 Existence of optimal solutions

We now formulate the optimal control problem to which this section is dedicated. It transpires that the continuous dependence results from Section 4.2 enable us to prove a theorem on the existence of optimal processes to some optimal control problem. Specifically, we shall consider control problem governed by boundary value problem (16) with the integral cost functional

$$(20) \quad J(u, v, w) = \int_{\Omega} \theta(x, u(x), (-\Delta)^{\alpha/4} u(x), v(x), (-\Delta)^{\alpha/4} v(x), w(x)) dx$$

where $\theta : \Omega \times \mathbb{R}^{4+m} \rightarrow \mathbb{R}$ is a given function. Here $(u, v) \in \mathbb{H}_0^{\alpha/2}$ is the trajectory and $w \in \mathcal{W}$ is the distributed control where

$$\mathcal{W} = \{w \in L^p(\Omega, \mathbb{R}^m) : w(x) \in M \text{ for a.e. } x \in \Omega\}$$

with $p \in \left(\frac{2n}{n+\alpha}, \infty\right]$ and M being a compact and convex subset of \mathbb{R}^m .

Let \mathcal{D} be the set of all admissible triples, i.e.

$$\mathcal{D} = \left\{ (u, v, w) \in \mathbb{H}_0^{\alpha/2} \times \mathcal{W} : (u, v) \text{ is a weak solution to (16) for } w \in \mathcal{W} \right\}.$$

It is worth noting that under assumptions of Theorem 3.2 the set of all admissible triples \mathcal{D} is nonempty, see Remark 3.1. In this section, our aim is to find a triple $(u_{w^*}, v_{w^*}, w^*) \in \mathcal{D} \subset \mathbb{H}_0^{\alpha/2} \times \mathcal{W}$ that minimizes the cost given by the functional (20), i.e. we look for a triple (u_{w^*}, v_{w^*}, w^*) satisfying

$$(21) \quad J(u_{w^*}, v_{w^*}, w^*) = \min_{(u, v, w) \in \mathcal{D}} J(u, v, w).$$

On the integrand θ from the cost functional defined in (20) we impose the following conditions:

(A6) the function $\theta = \theta(x, u, p, v, q, w)$ is measurable with respect to x for all $(u, p, v, q, w) \in \mathbb{R}^4 \times M$ continuous with respect to (u, p, v, q, w) for a.e. $x \in \Omega$ and convex with respect to w for all $(u, p, v, q) \in \mathbb{R}^4$ and a.e. $x \in \Omega$. Moreover, there exists a constant $c > 0$ such that

$$|\theta(x, u, p, v, q, w)| \leq c \left(1 + |u|^s + |p|^2 + |v|^s + |q|^2 \right)$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}$, $p \in \mathbb{R}$, $v \in \mathbb{R}$, $q \in \mathbb{R}$, $w \in M$ and for some $s \in (1, 2_\alpha^*)$;

(A7) there exist a function $\eta \in L^1(\Omega)$ and a constant $C > 0$ such that

$$\theta(x, u, p, v, q, w) \geq \eta(x) - C(|u| + |p| + |v| + |q| + |w|)$$

for all $u \in \mathbb{R}$, $p \in \mathbb{R}$, $v \in \mathbb{R}$, $q \in \mathbb{R}$, $w \in M$ and a.e. $x \in \Omega$.

Now we prove a theorem on the existence of optimal processes to our optimal control problem (21).

Theorem 5.1 *If the function G of the form (15) satisfies (A1'), (A2'), (A3), (A4), (A5) and the integrand θ meets assumptions (A6), (A7), then the optimal control problem (21) possesses at least one optimal process (u_{w^*}, v_{w^*}, w^*) .*

Proof. From (A6), (A7) and classical theorems on semicontinuity of integral functional see among others [6, Theorem 1], [35, Theorem 1.1] or [29, Theorem 5], we deduce that J is lower semicontinuous with respect to the strong topology in the space $\mathbb{H}_0^{\alpha/2}$ and either the weak topology of $L^p(\Omega, \mathbb{R}^m)$ for $p \in \left(\frac{2n}{n+\alpha}, \infty\right)$ or the weak * topology of $L^\infty(\Omega, \mathbb{R}^m)$, since convergence of any sequence $\{u_k\}_{k \in \mathbb{N}}$ in $H_0^{\alpha/2}(\Omega)$ implies the strong convergence of $\{u_k\}_{k \in \mathbb{N}}$ in $L^s(\Omega)$ with $s \in (1, 2_\alpha^*)$ and the strong convergence of $\{(-\Delta)^{\alpha/4} u_k\}_{k \in \mathbb{N}}$ in $L^2(\Omega)$ and moreover we have the same implications for convergence of any sequence $\{v_k\}_{k \in \mathbb{N}}$ in $H_0^{\alpha/2}(\Omega)$.

Next, let $\{(u_k, v_k, w_k)\}_{k \in \mathbb{N}} \subset \mathcal{D}$ be a minimizing sequence for optimal control problem (21), i.e.

$$(22) \quad \lim_{k \rightarrow \infty} J(u_k, v_k, w_k) = \inf_{(u, v, w) \in \mathcal{D}} J(u, v, w) = \vartheta.$$

Since the set M is compact and convex, we see that the sequence $\{w_k\}_{k \in \mathbb{N}}$ is compact in the weak topology of $L^p(\Omega, \mathbb{R}^m)$ for $p \in \left(\frac{2n}{n+\alpha}, \infty\right)$ or the weak * topology of $L^\infty(\Omega, \mathbb{R}^m)$, respectively. Passing to subsequence if necessary, one can assume that w_k tends to some $w_0 \in \mathcal{W}$ weakly in $L^p(\Omega, \mathbb{R}^m)$ or w_k tends to some $w_0 \in \mathcal{W}$ weakly * in $L^\infty(\Omega, \mathbb{R}^m)$, respectively. By assumption (A5), the set of the weak solutions of problem (16) coincides with the set of saddle points of the functional F_{w_k} on the space $\mathbb{H}_0^{\alpha/2}$, see Remark 3.1. By Propositions 4.4 or 4.5, the sequence $\{(u_k, v_k)\}_{k \in \mathbb{N}}$, or at least some its subsequence, tends to (u_0, v_0) in $\mathbb{H}_0^{\alpha/2}$

and the triple (u_0, v_0, w_0) is an admissible triple for control problem (16). Due to the lower semicontinuity of J , we have

$$(23) \quad J(u_0, v_0, w_0) \leq \liminf_{k \rightarrow \infty} J(u_k, v_k, w_k)$$

provided (u_k, v_k) tends to (u_0, v_0) in $\mathbb{H}_0^{\alpha/2}$ and $w_k \rightharpoonup w_0$ weakly in $L^p(\Omega, \mathbb{R}^m)$ or $w_k \xrightarrow{*} w_0$ weakly $*$ in $L^\infty(\Omega, \mathbb{R}^m)$, respectively. Furthermore, by (22) and (23), we have

$$\vartheta \leq J(u_0, v_0, w_0) \leq \liminf_{k \rightarrow \infty} J(u_k, v_k, w_k) = \inf_{(u,v,w) \in \mathcal{D}} J(u, v, w) = \vartheta.$$

Thus, $J(u_0, v_0, w_0) = \vartheta = \inf_{(u,v,w) \in \mathcal{D}} J(u, v, w)$. It means that the process $(u_{w^*}, v_{w^*}, w^*) = (u_0, v_0, w_0)$ is optimal for (21). ■

Remark 5.1 From the proof of Theorem 5.1 one can see that it suffices to assume weaker assumption on controls than M to be compact and convex, namely only boundedness of w_k in $L^p(\Omega, \mathbb{R}^m)$.

Remark 5.2 By a direct calculation, one can check that the quadratic functional

$$\mathcal{F}(u, v) = \frac{1}{2} \int_{\Omega} \left(\left| (-\Delta)^{\alpha/4} v(x) \right|^2 - \xi_1 |v(x)|^2 - \left| (-\Delta)^{\alpha/4} u(x) \right|^2 + \xi_2 |u(x)|^2 \right) dx$$

is strictly concave in u and strictly convex in v for $\xi_1 < \rho_1^{\alpha/2}$, $\xi_2 < \rho_1^{\alpha/2}$ and concave in u and convex in v for $\xi_1 = \xi_2 = \rho_1^{\alpha/2}$ where ρ_1 is the principal eigenvalue of the operator $-\Delta$ defined on $H_0^1(\Omega)$.

Since

$$F_w(u, v) = \mathcal{F}(u, v) + \int_{\Omega} \left(\frac{\xi_1}{2} |v(x)|^2 - \frac{\xi_2}{2} |u(x)|^2 + G^1(x, u(x), v(x)) + G^2(x, u(x), v(x)) w(x) \right) dx$$

$(u, v) \in \mathbb{H}_0^{\alpha/2}$. From the remark mentioned above, it can be easily seen that assumption (A5) can be relaxed as stated in the next corollary. In fact, Theorem 5.1 implies:

Corollary 5.1 The optimal control problem (21) possesses at least one optimal process (u_{w^*}, v_{w^*}, w^*) provided the function G of the form (15) satisfies (A1'), (A2'), (A3), (A4) the integrand θ meets assumptions (A6), (A7) and the function $\frac{\xi_1}{2} |v|^2 - \frac{\xi_2}{2} |u|^2 + G^1(x, u, v) + G^2(x, u, v) w$ is concave in u and convex in v for some $\xi_1 < \rho_1^{\alpha/2}$, $\xi_2 < \rho_1^{\alpha/2}$, all $w \in \mathcal{W}$ and a.e. $x \in \Omega$.

Remark 5.3 Let us consider a fractional differential equation of the form

$$(24) \quad \begin{cases} (-\Delta)^{\alpha/2} v(x) + G_v(x, v(x), w(x)) = 0 & \text{in } \Omega \\ v(x) = 0 & \text{on } \partial\Omega \end{cases}$$

with the functional of action

$$(25) \quad F_w(v) = \int_{\Omega} \left(\frac{1}{2} \left| (-\Delta)^{\alpha/4} v(x) \right|^2 + G(x, v(x), w(x)) \right) dx$$

where $v \in H_0^{\alpha/2}(\Omega)$ and $w \in \mathcal{W}$. Problem (24) is a particular case of problem (6) as one can consider problem (6) without u . Clearly, under assumptions (A1), (A2), (A3), (A5) where we omit the variable u , the functional of action in (25) is coercive on $H_0^{\alpha/2}(\Omega)$ and all the results of this paper are valid for the functional (25) and problem (24). For related results on the problem (24) with the Dirichlet fractional Laplacian, see [13].

6 Examples

Example 6.1 Let Ω be a cube of the form

$$\Omega = P^3(0, \pi) = \{x \in \mathbb{R}^3 : 0 < x_i < \pi, i = 1, 2, 3\}.$$

Note that $u_1 = \sin x_1 \sin x_2 \sin x_3$ and $\rho_1 = 3$ are eigenfunction and eigenvalue for $-\Delta$ on $H_0^1(\Omega)$ since $-\Delta u_1 = 3u_1$. Similarly, $(-\Delta)^{\alpha/2} u_1 = 3^{\alpha/2} u_1$ hence, $3^{\alpha/2}$ is the first eigenvalue for $(-\Delta)^{\alpha/2}$ in this case. Consider the following linear control problem involving the fractional Laplacian

$$(26) \quad \begin{cases} -(-\Delta)^{\alpha/2} u(x) + \beta_1 u(x) + w_1(x) v(x) + l_1(x) = 0 & \text{in } \Omega \\ (-\Delta)^{\alpha/2} v(x) - \beta_2 v(x) + w_2(x) u(x) + l_2(x) = 0 & \text{in } \Omega \\ u(x) = 0, v(x) = 0 & \text{on } \partial\Omega \end{cases}$$

where $\beta_i < 3^{\alpha/2}$, $l_i \in L^2(\Omega)$ for $i = 1, 2$ and

$$\mathcal{W} = \{w \in L^p(\Omega, \mathbb{R}^2) : w(x) \in [0, 1] \times [0, 1] \text{ for a.e. } x \in \Omega\}$$

with $p \in (\frac{3}{\alpha}, \infty)$. The functional of action for control problem (26) is of the form

$$\begin{aligned} F_w(u, v) = \int_{\Omega} & \left(\frac{1}{2} \left| (-\Delta)^{\alpha/4} v(x) \right|^2 - \frac{1}{2} \left| (-\Delta)^{\alpha/4} u(x) \right|^2 + \frac{\beta_1}{2} |u(x)|^2 - \frac{\beta_2}{2} |v(x)|^2 \right. \\ & \left. + (w_1(x) + w_2(x)) u(x) v(x) + l_1(x) u(x) + l_2(x) v(x) \right) dx. \end{aligned}$$

It is easily checked that the function

$$G(x, u, v, w) = \frac{\beta_1}{2} u^2 - \frac{\beta_2}{2} v^2 + (w_1 + w_2) uv + l_1(x) u + l_2(x) v$$

satisfies assumption (A1), growth conditions (A2) and (A3), (A4) with $b = \frac{\beta_1}{2}$, $B = \frac{\beta_2}{2}$, respectively. Moreover, the functional F_w is strictly concave in u and strictly convex in v . Thus, for any $w_k \in \mathcal{W}$, there exists a unique weak solution (u_k, v_k) of control problem (26), cf. Theorem 3.2 and Remark 3.1. If w_k tends to w_0 in the strong topology of $L^p(\Omega, \mathbb{R}^2)$ with $p \in (\frac{3}{\alpha}, \infty)$, then (u_k, v_k) tends to (u_0, v_0) in $\mathbb{H}_0^{\alpha/2}$, cf. Proposition 4.3. Since $\frac{3}{\alpha} > \frac{6}{3-\alpha}$, we can also consider either the weak convergence of controls in $L^p(\Omega, \mathbb{R}^2)$ with $p \in (\frac{6}{3-\alpha}, \infty)$ as stated in Proposition 4.4 or the weak $*$ convergence of controls in $L^\infty(\Omega, \mathbb{R}^2)$ as stated in Proposition 4.5 to get the same strong convergence of weak solutions. Moreover, one can check that there exists an optimal control w^* such that the triple (u_{w^*}, v_{w^*}, w^*) is an admissible triple for the process described by control problem (26) minimizing the cost functional of the form

$$J(u, v, w) = \int_{\Omega} \left(u^s(x) + v^s(x) + |w(x)|^2 \right) dx$$

where $s \in (1, \frac{6}{3-\alpha})$, cf. Theorem 5.1.

Example 6.2 Let $\Omega = P^3(0, \pi)$ be a cube as in Example 6.1. The control problem now is of the form

$$(27) \quad \begin{cases} -(-\Delta)^{\alpha/2} u(x) + bu(x) - s|x|^2 u^{s-1}(x) w_1(x) - |x| w_2(x) + v(x) = 0 & \text{in } \Omega \\ (-\Delta)^{\alpha/2} v(x) - av(x) + s|x|^2 v^{s-1}(x) w_1(x) - |x| w_2(x) + u(x) = 0 & \text{in } \Omega \\ u(x) = 0, v(x) = 0 & \text{on } \partial\Omega \end{cases}$$

for $1 + \frac{1}{p-1} < s < \frac{6}{3-\alpha}$ with $p \in (\frac{6}{3+\alpha}, \infty)$ or $1 < s < \frac{6}{3-\alpha}$ with $p = \infty$. Now, the cost is given by

$$(28) \quad J(u, v, w) = \int_{\Omega} \left[u^s(x) + \left| (-\Delta)^{\alpha/4} u(x) \right|^2 w_1(x) + \left| (-\Delta)^{\alpha/4} v(x) \right|^2 w_2(x) - |x| \left| (-\Delta)^{\alpha/4} u(x) + |w(x)|^2 \right| \right] dx$$

where $a < 3^{\alpha/2}$, $b < 3^{\alpha/2}$ and $M = [0, 1] \times [0, 1]$. Obviously, the functional of action for control problem (27) has the form

$$F_w(u, v) = \int_{\Omega} \left[\frac{1}{2} \left| (-\Delta)^{\alpha/4} v(x) \right|^2 - \frac{1}{2} \left| (-\Delta)^{\alpha/4} u(x) \right|^2 - \frac{a}{2} v^2(x) + \frac{b}{2} u^2(x) + |x|^2 v^s(x) w_1(x) - |x|^2 u^s(x) w_1(x) - u(x) |x| w_2(x) - v(x) |x| w_2(x) + u(x) v(x) \right] dx.$$

It is easy to check that the functionals F_w and J satisfy all assumptions of Theorems 3.2 and 5.1. By Remark 5.2, F_w is strictly concave in u and strictly convex in v . Thus, Theorem 3.2 and Remark 3.1 imply that for any control w there exists exactly one weak solution (u_w, v_w) of control problem (27) and moreover from Propositions 4.4 and 4.5 one can deduce that the weak solution continuously depends on control w provided controls converge in the weak topology of $L^p(\Omega, \mathbb{R}^2)$ with $p \in \left(\frac{6}{3-\alpha}, \infty\right)$ or the weak $*$ topology of $L^\infty(\Omega, \mathbb{R}^2)$, respectively. Moreover, from Theorem 5.1, we infer that there exists an optimal control w^* such that the triple (u_{w^*}, v_{w^*}, w^*) is an admissible triple of the process described by control problem (27) that minimizes the cost functional given by (28).

7 Concluding remarks

As far as we know, the question of the continuous dependence on functional parameters or controls of the solutions of control problem governed by fractional differential equations involving the Dirichlet fractional Laplacian has not been considered up to now. In this paper, we have established the existence and continuous dependence on the functional parameter of weak solutions corresponding to saddle critical points of the functional of action. Furthermore, the existence of optimal processes minimizing the cost functional was ascertained. The novelty of the results lies in the nonlocal structure of both action and cost functionals depending on the values of nonlocal Dirichlet fractional Laplace operator.

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